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The clock of a quantum computer

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Abstract

If the physical agent (the 'pointer', or 'cursor', or 'clocking mechanism') that sequentially scans the T lines of a long computer program is a microscopic system, two quantum phenomena become relevant: spreading of the probability distribution of the pointer along the program lines, and scattering of the probability amplitude at the two endpoints of the physical space allowed for its motion.

We show that the first effect determines an upper bound $O\left(T^{-\frac{2}{3}}\right)$ on the probability of finding the pointer exactly at the END line.

By adding an adequate number δ of further empty lines ('telomers'), one can store the result of the computation up to the moment in which the pointer is scattered back into the active region. This leads to a less severe upper bound $O(\sqrt{\delta/T}\,)$ on the probability of finding the pointer either at the END line or within the additional empty lines.

Our analysis is performed in the context of Feynman's model of quantum computation, the only model, to our knowledge, that explicitly includes a physically plausible quantum clocking mechanism in its considerations.

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1. Introduction

It has been observed by Margolus [1] that Feynman [2], in his model of a quantum computer, 'managed to arrange for all the quantum uncertainty [. . .] to be concentrated in the time taken for the computation to be completed, rather than in the correctness of the answer'. In this paper we try to contribute to a quantitative assessment of this time uncertainty.

Recent work by Levitin and Margolus [3] relates the maximum rate of information processing by a quantum computer to the available, *conserved*, energy. It is therefore of some theoretical interest to revisit a model, such as Feynman's, based on a closed system, evolving according to a time-independent and, therefore, *conserved* Hamiltonian.

The doubt has been raised by Alicki [4] that 'the idea that the physical time [. . .] of computation is proportional to the complexity [. . .] is only true for the existing digital computers which are ensembles of controlled bistable elements which [. . .] can literally mimic logical operations'.

Feynman's model provides an ideal context for the study of this issue: timing is modelled by a cursor, which jumps along a sequence of sites, indicating that the corresponding discrete operations should be applied. The cursor itself is treated as a quantum dynamical system, which imposes limitations on our ability to know, without performing a measurement, *whether* the computation has finished. We show that in this model, there is a 'probabilistic time overhead' due to the fact that at *no* instant of time the probability that the computation turns out to be completed is equal to 1.

In Feynman's model, the sequential application, in the correct order, of T reversible primitives to the input/output register requires the introduction of at least $x_0 = T + 1$ program counter sites. At the beginning of a computation an assigned input is written on the input/output register, and the cursor is placed at site number 1; as the cursor jumps from site j to site $j + 1$, the jth primitive is applied. If at any time t the cursor is found, as the result of the first measurement of its position, to be at site x_0, then the desired output turns out to be written on the input/output register.

We call $Q(t)$ the position of the cursor at time t; the issue of optimally choosing the time $t_{\text{optimal}}(x_0)$ at which the first position measurement is performed, in order to maximize the probability $P(Q(t) = x_0)$, has been discussed by Gramß [5]. It turns out that with a convenient choice of the unit of time,

$$t_{\text{optimal}}(x_0) = x_0 + \text{const}\; x_0^{\frac{1}{3}}. \tag{1.1}$$

Calling $p_1(x_0)$ the maximum of $P(Q(t) = x_0)$ attained at this instant, we wish to show that

$$p_1(x_0) \leqslant \frac{8}{x_0^{\frac{2}{3}}}. \tag{1.2}$$

This dismayingly severe upper bound can be somewhat relaxed if one accepts Feynman's suggestion [2] of improving the action of his computer by the addition of a 'telomeric' chain of δ sites on the right of site x_0: during its jumps between sites of the telomeric chain, the cursor keeps applying the identity primitive, thus, in effect, storing the result of the computation until it is reflected back into the non-telomeric region.

We call $p_1(x_0, \delta)$ the probability of finding, at an optimally chosen time $t_{\text{optimal}}(x_0, \delta)$, the cursor in $\{x_0, \dots, x_0 + \delta\}$ and, therefore, of finding that the computation has been completed. We show that, for $x_0 \gg 1$ and $\delta \gg 1$, $p_1(x_0, \delta)$ satisfies the approximate equality

$$p_1(x_0, \delta) \approx 1 - \frac{2}{\pi}\left(\arcsin\left(\frac{1}{1 + 2\delta/x_0}\right) - \left(\frac{1}{1 + 2\delta/x_0}\right)\sqrt{1 - \left(\frac{1}{1 + 2\delta/x_0}\right)^2}\right). \tag{1.3}$$

For small values of the ratio δ/x_0 one can also write the more expressive approximate equality

$$p_1(x_0, \delta) \approx \frac{8}{\pi}\sqrt{\frac{\delta}{x_0}}. \tag{1.4}$$

The message carried by (1.2) and (1.3) is best illustrated by a numerical example. Suppose the computation at hand requires a number of steps such that $x_0 \approx 10^9$; inequality (1.2) says that in the absence of telomers, at *no* instant of time the probability of finding the computation completed is larger than 10^{-5} (for short, reversible quantum computation with a quantum clocking mechanism is practically impossible without telomers); the approximate equality (1.3) says, in turn, that the probability of finding the computation completed can, at a suitably chosen instant of time, be made close to 1 provided the ratio δ/x_0 is large enough: for the

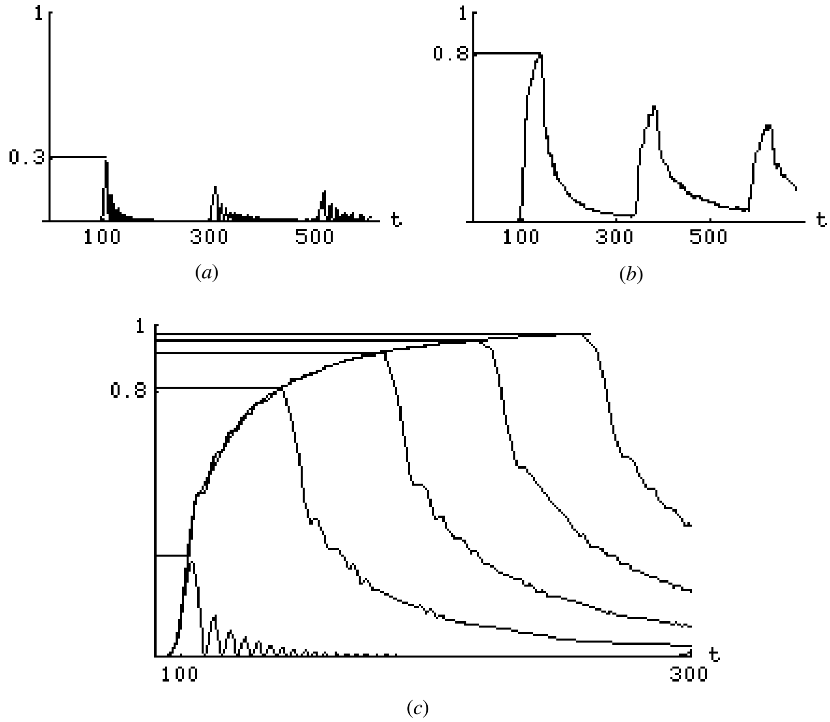


Figure 1. $P(x_0 \leqslant Q(t) \leqslant x_0 + \delta)$ as a function of time, computed using Gramß' explicit solution of the Schrödinger equation for the Feynman quantum computer. The horizontal lines are the upper bounds obtained in the following sections. In all frames $x_0 = 100$. (*a*) $\delta = 0$; (*b*) $\delta = 20$; (*c*) the initial parts of frames (*a*) and (*b*) are here shown together with the similar graphs corresponding to the cases $\delta = 40, 60, 80$.

numerical example at hand this probability can be made close to 0.9 for $\delta \approx 3.4 \times 10^8$, or close to 0.99 for $\delta \approx 1.2 \times 10^9$ (short, reversible quantum computation with a quantum clocking mechanism is practically possible provided sufficient *space* resources are available). Further examples of the meaning of (1.2) and (1.3) are summarized in figure 1.

Before going into the asymptotic estimates for large values of x_0 leading to (1.2) and (1.3) we wish to observe that the phenomenon studied in this paper (the uncertainty on *whether* the computation has finished) already presents itself in such an elementary task as the implementation of the CNOT gate. We examine this case in all detail in order, mainly, to give a precise physical idea of the computer architecture to which our considerations apply.

We use the following notation for the CNOT function:

$$\mathrm{CNOT} : (z_1, z_2) \in \{-1, 1\}^2 \to \big(z_1, \big(1 - 2\delta_{z_1,1}\big) z_2\big) \tag{1.5}$$

('flip the controlled bit z_2 if and only if the controlling bit z_1 has the value +1').

Feynman [2] proposed the following Hamiltonian, diagrammatically represented in figure 2, for the quantum reversible implementation of the CNOT primitive:

$$\begin{aligned} H_{\mathrm{CNOT}} = {} & \sigma_-(1)\tau_-(1)\tau_+(2) + \sigma_x(2)\tau_-(2)\tau_+(3) + \sigma_+(1)\tau_-(3)\tau_+(6) + \sigma_+(1)\tau_-(1)\tau_+(4) \\ & + I\,\tau_-(4)\tau_+(5) + \sigma_-(1)\tau_-(5)\tau_+(6) + \text{Hermitian conjugate}. \end{aligned} \tag{1.6}$$

The six spin-1/2 systems $\underline{\tau}(1), \underline{\tau}(2), \ldots, \underline{\tau}(6)$, forming the 'cursor', play here the role of a clocking and synchronization apparatus, scaled down to the quantum regime.

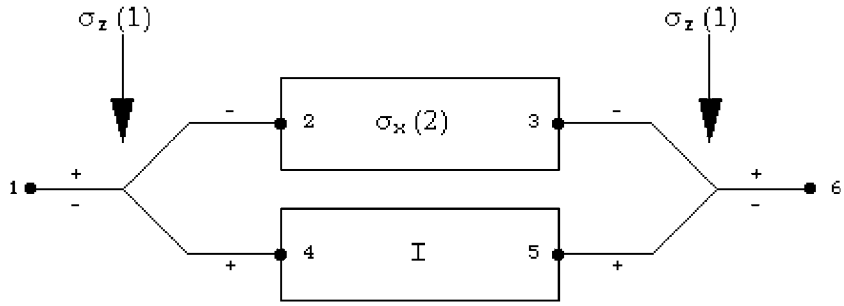


Figure 2. This figure is a reproduction of figure 8 of [2], adapted to our notations.

The operators $\tau_\pm(j) = (\tau_x(j) \pm \mathrm{i}\tau_y(j))/2$ are the raising and lowering operators for the z-components of the cursor spins.

The two spin-1/2 systems $\underline{\boldsymbol{\sigma}}(1)$, $\underline{\boldsymbol{\sigma}}(2)$ constitute the 'register', $\underline{\boldsymbol{\sigma}}(1)$ being the controlling q-bit, $\underline{\boldsymbol{\sigma}}(2)$ the controlled one. The operators $\sigma_\pm(j) = (\sigma_x(j) \pm \mathrm{i}\sigma_y(j))/2$ are the raising and lowering operators for the z-components of the register spins.

The operator $N = \sum_{k=1}^{6}(1 + \tau_3(k))/2$ commutes with H_{CNOT}. By a suitable choice of the initial condition, we are going to restrict our considerations to the eigenspace belonging to the eigenvalue 1 of N. It is, in this subspace, useful to introduce the operator

$$Q = \sum_{k=1}^{6} k \frac{1 + \tau_3(k)}{2} \tag{1.7}$$

to be thought as the 'position operator' for the cursor.

Figure 2 provides the basic intuition for understanding the time evolution under H_{CNOT} of an initial condition being a simultaneous eigenstate of $\sigma_z(1)$, $\sigma_z(2)$ and Q, belonging to the eigenvalue 1 of Q: it calls our attention to the following four possible classical computational paths:

$$\begin{cases} \Gamma_{(+1,+1)} = (((+1,+1),1),((-1,+1),2),((-1,-1),3),((+1,-1),6)) \\ \Gamma_{(+1,-1)} = (((+1,-1),1),((-1,-1),2),((-1,+1),3),((+1,+1),6)) \\ \Gamma_{(-1,+1)} = (((-1,+1),1),((+1,+1),4),((+1,+1),5),((-1,+1),6)) \\ \Gamma_{(-1,-1)} = (((-1,-1),1),((+1,-1),4),((+1,-1),5),((-1,-1),6)). \end{cases} \tag{1.8}$$

For an intuitive motivation of the interest of these paths we refer the reader to Feynman's presentation of figure 8 of [2].

The path $\Gamma_{(+1,+1)}$, for example, is to be read in the following way:

- start at $\Gamma_{(+1,+1)}(1) = ((+1,+1),1)$, with both register spins 'up' and cursor at position 1;
- move to $\Gamma_{(+1,+1)}(2) = ((-1,+1),2)$, with the controlling spin flipped and cursor at position 2;
- move to $\Gamma_{(+1,+1)}(3) = ((-1,-1),3)$, with controlled spin flipped and cursor at position 3;
- stop at $\Gamma_{(+1,+1)}(4) = ((+1,-1),6)$, with controlling spin restored to the initial value and cursor at position 6.

Note that the paths starting with the controlling spin 'down' take, instead, the lower route in figure 2. In this lower route, as compared with the upper route, the NOT primitive $\sigma_x(2)$ is substituted by the identity primitive I: this plays the role of a delay line, having the role of making all the computational paths of the same length 4.

For each computational path, at step 4, *and only at that step*, the content of the register is the CNOT of its content at step 1.

The semiclassical intuition developed up to this point is confirmed by the explicit solution of the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -\frac{\lambda}{2}H_{\mathrm{CNOT}}|\psi(t)\rangle \tag{1.9}$$

(where we have, for convenience, written a coupling constant as $-\lambda/2$) under an initial condition of the form

$$|\psi(0)\rangle = |\sigma_z(1) = z_1, \sigma_z(2) = z_2, Q = 1\rangle \equiv |((z_1, z_2), 1)\rangle. \tag{1.10}$$

Such a solution can be written in the form

$$|\psi(t)\rangle = \sum_{j=1}^{4} \gamma(t, j; 4)\left|\Gamma_{(z_1,z_2)}(j)\right\rangle \tag{1.11}$$

where the amplitudes γ do not depend on the initial condition (z_1, z_2) imposed on the z-components of the register spins and depend on the computation being performed only through the length (4 in the example at hand) of the computational paths.

From the form of solution (1.11) and from the fact that only at step 4 the content of the register is the CNOT of its content at step 1, we see that at every time $t > 0$ the *conditional* probability, *given* that a measurement of Q has given the value 4, that a measurement of the z-components of the two register spins gives, respectively, $z_1' = z_1$ and $z_2' = \left(1 - 2\delta_{z_1,1}\right)z_2$ is equal to 1: there is no uncertainty in the correctness of the answer.

The only uncertainty is in the probability $|\gamma(t, 4; 4)|^2$ of finding the cursor in position 4. We claim that for no value of t this probability takes the value 1.

Indeed it can be shown, by explicit integration of (1.9) and (1.10), that

$$\gamma(t, 4; 4) = \mathrm{i}\left(\cos\left(\frac{\lambda t\sqrt{5}}{4}\right)\sin\left(\frac{\lambda t}{4}\right) - \frac{1}{\sqrt{5}}\cos\left(\frac{\lambda t}{4}\right)\sin\left(\frac{\lambda t\sqrt{5}}{4}\right)\right). \tag{1.12}$$

Therefore $|\gamma(t, 4; 4)|^2$ reaches its local maxima at the points $t_k = 4\pi k/(\lambda\sqrt{5})$, k being an integer, where it takes the values

$$|\gamma(t_k, 4; 4)|^2 = \sin^2\left(\frac{k\pi}{\sqrt{5}}\right) < 1. \tag{1.13}$$

We observe, incidentally, that while inequality (1.2) becomes trivial for $x_0 < 23$, inequality (1.13) gives an example of the possibility, for such small values of x_0, of proving more stringent upper bounds, without making recourse to asymptotic estimates.

Gramß [6] gave numerical and graphical evidence of the fact that the maximum of the probability to observe the completion of a computation on a Feynman computer decreases when the number of elementary gates increases. The issue of this paper is to give upper bounds and asymptotic estimates for this probability.

The paper is organized as follows. In section 2 we review Gramß' solution [5] of the Schrödinger equation for the Feynman quantum computer and, by the simple artifice of studying its limit as the number of sites tends to infinity, we single out the physical effect which is at the root of the above inequalities: the spreading of the wave packet describing the quantum motion of the cursor. In section 3 we deal with the technicalities involved in reinstating the reflecting boundary condition on the rightmost site of a finite chain of program counter sites. Section 4 addresses the issue of the comparison with a 'standard' model of quantum computation, such as that of quantum circuits, and discusses the consequences of the upper bounds (1.2)–(1.4) on some quantum algorithms.

Some technical points have been collected in the appendices.

2. The semi-infinite clock

It is important to emphasize that, in Feynman's architecture, the motion of the cursor is independent of the computation being performed.

This fact has been observed in [2, 7] and is reviewed in some detail in appendix A for the case in which the motion of the cursor is restricted to jumps between nearest-neighbour sites.

The example of the CNOT primitive given in the introduction, easily extended to the Toffoli primitive, shows that the above statement extends to the case in which conditional jumps to non-nearest-neighbour sites are required, provided all computational paths are made of the same length by the introduction of suitable delay lines or, equivalently stated in Feynman's suggestive terminology, by suitable 'impedance matching in transmission lines'.

The motion of the cursor along a computational path is described (calling $\Lambda_s = \{1, 2, \ldots, s\}$ the finite collection of sites on which it takes place) by the following Schrödinger equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t, x; s) = -\frac{1}{2}(\psi(t, x-1; s) + \psi(t, x+1; s)) \qquad x \in \Lambda_s \tag{2.1}$$

$$\psi(t, 0; s) = 0 \tag{2.2}$$

$$\psi(t, s+1; s) = 0. \tag{2.3}$$

We refer the reader to [8] for a discussion of the relevance of equation (2.1) on an infinite lattice (without, namely, the boundary conditions (2.2) and (2.3) which play a prominent role in our analysis) to the contemporary theory of continuous time quantum random walks.

With an eye to a conceivable physical implementation of a quantum clocking mechanism, we wish to recall that (2.1) was proposed by Feynman himself [9] as a model of the motion of excitations along a one-dimensional crystal.

A great deal of physical insight about the system described by (2.1)–(2.3) can be gained from the Ehrenfest equation of motion: the expectation value of the position of the cursor

$$\mu(t; s) = \sum_{x=1}^{s} x|\psi(t, x; s)|^2 \tag{2.4}$$

satisfies the equation

$$\frac{\mathrm{d}^2\mu(t; s)}{\mathrm{d}t^2} = \frac{1}{2}(|\psi(t, 1; s)|^2 - |\psi(t, s; s)|^2). \tag{2.5}$$

The mean motion of the cursor is inertial (after all, (2.1) is a lattice version of the free Schrödinger equation) as long as there is no significant unbalance between the probability of finding it in the initial site 1 and the probability of finding it in the final site s.

Equations (2.1)–(2.3) have been extensively studied by Gramß [5]. In particular, their solution under the initial condition

$$\psi(0, x; s) = \delta_{1,x} \qquad x \in \Lambda_s \tag{2.6}$$

is explicitly given by

$$\psi(t, x; s) = \frac{2}{s+1}\sum_{n=1}^{s} \exp[\mathrm{i}t\cos(\vartheta(n; s))]\sin(\vartheta(n; s))\sin(x\vartheta(n; s)) \tag{2.7}$$

where

$$\vartheta(n; s) = \frac{n\pi}{s+1}. \tag{2.8}$$

In order to gain some intuition and to collect some mathematical facts about the behaviour of solution (2.7), (2.8), we study, in this section, its limit as $s \to +\infty$. We shall refer to this limit situation as to the case of a 'semi-infinite clock'. The infinite chain of additional sites appended to the right of the cursor plays, in this section, the purely technical role of making the analysis simpler (in much the same way as a semi-infinite lines of nodes appended at the starting node of a quantum decision tree is technically expedient in Farhi and Gutmann's work [10]); the effects of reflection at the rightmost site of a *finite* chain will be considered in the next section.

Routine manipulations of the integral representation of the Bessel coefficients $J_x(t)$ [11] yield the following expression for the limit of $\psi(t, x; s)$ as $s \to +\infty$:

$$\varphi(t, x) \equiv_{\text{def}} \lim_{s \to +\infty} \psi(t, x; s) = i^{x-1}(J_{x-1}(t) + J_{x+1}(t)) = i^{x-1}\frac{2x}{t}J_x(t). \tag{2.9}$$

This can also be directly checked by observing that, because of the recurrence relations for the Bessel functions, $i^{x-1}(J_{x-1}(t) + J_{x+1}(t))$ is the solution of (2.1) under the single boundary condition (2.2) and the initial condition (2.6).

Some elementary properties of the Bessel functions and their role in deriving the results of this section and section 3 are, for the convenience of the reader, summarized in appendix B.

In this section we study some probabilistic aspects of the random variable $Q(t)$ giving the position of the cursor in the state $\varphi(t, x)$, subject to the probability law

$$f(t, x) \equiv_{\text{def}} P(Q(t) = x) = \frac{4x^2}{t^2}J_x(t)^2 \qquad x = 1, 2, \ldots. \tag{2.10}$$

The expectation value

$$\mu(t) = E(Q(t)) \tag{2.11}$$

can be computed by the observation that, because of (2.5), it satisfies the differential equation

$$\frac{\mathrm{d}^2\mu(t)}{\mathrm{d}t^2} = \frac{2}{t^2}J_1(t)^2. \tag{2.12}$$

As, at time $t = 0$, $\mu(t)$ reaches its absolute minimum equal to 1, the above equation is to be solved under the initial condition

$$\mu(0) = 1 \qquad \mu'(0) = 0. \tag{2.13}$$

The solution of (2.12) and (2.13) is a generalized hypergeometric function [12], which, in the notation of [13], can be written as

$$\mu(t) = \text{HypergeometricPFQ}[\{-1/2\}, \{1, 2\}, -t^2]. \tag{2.14}$$

The second moment $E(Q(t)^2)$ can be easily computed using the series expansion of t^4 in terms of Bessel coefficients [11]. We then get

$$E(Q(t)^2) = 1 + \tfrac{3}{4}t^2. \tag{2.15}$$

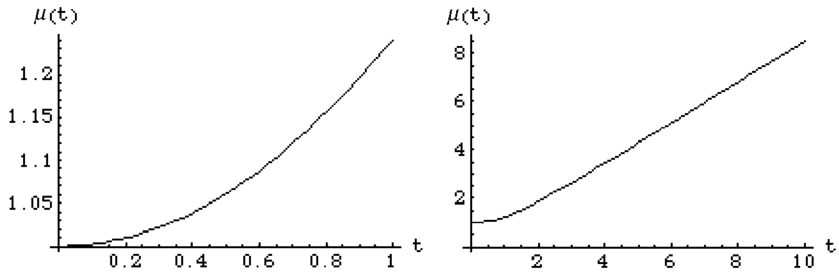
The variance of $Q(t)$ is then easily computed as

$$\text{var}(Q(t)) = E(Q(t)^2) - E(Q(t))^2. \tag{2.16}$$

The results (2.14) and (2.16) are summarized in figures 3 and 4, respectively.

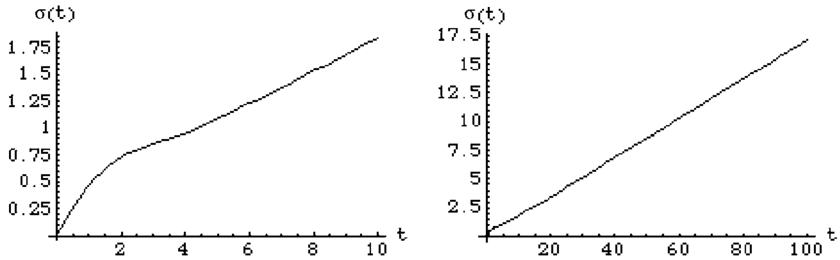
It is not easy to write in a closed form the cumulative distribution function

$$F(t, x) \equiv_{\text{def}} P(Q(t) \leqslant x) = \frac{4}{t^2}\sum_{1 \leqslant k \leqslant x} k^2 J_k(t)^2 \tag{2.17}$$



(*a*) (*b*)

Figure 3. The behaviour of $\mu(t)$ as seen on two different time ranges. (*a*) For small values of t there is a mean acceleration due to the 'osmotic pressure' corresponding to the fact that there is a non-negligible probability mass at point $x = 1$. (*b*) For large values of t the mean motion of the cursor is inertial, to a high degree of approximation: $E(Q(t)) \approx \mu_1 t$, with $\mu_1 = 0.849$.



(*a*) (*b*)

Figure 4. After an initial transient (frame (*a*)), the standard deviation $\sigma(t) \equiv_{\mathrm{def}} \sqrt{\mathrm{var}(Q(t))}$ grows, to a high degree of approximation, linearly. On the time scale of frame (*b*), $\sigma(t) \approx \sigma_1 t$, with $\sigma_1 = 0.172$.

of the random variable $Q(t)$. Precisely because of the wave packet spreading observed in figures 3 and 4, it is, however, easy to study the limit in law of the standardized random variable

$$Q^*(t) \equiv_{\mathrm{def}} \frac{Q(t) - \mu(t)}{\sigma(t)}. \tag{2.18}$$

The limit as $t \to +\infty$ of the cumulative distribution function $P(Q^*(t) \leqslant z)$ turns out to be, setting $y(z) = \mu_1 + \sigma_1 \cdot z$,

$$F^*(z) \equiv_{\mathrm{def}} \lim_{t \to +\infty} P(Q^*(t) \leqslant z) = \begin{cases} \text{if} \quad z < -\dfrac{\mu_1}{\sigma_1} & \text{then } 0 \\ \text{if} \quad z > \dfrac{1 - \mu_1}{\sigma_1} & \text{then } 1 \\ \text{else} \quad \dfrac{2}{\pi}\left(\arcsin y(z) - y(z)\sqrt{1 - y(z)^2}\right) & \end{cases} \tag{2.19}$$

where

$$\mu_1 = \frac{8}{3\pi} \qquad \sigma_1 = \sqrt{\frac{3}{4} - \mu_1^2}. \tag{2.20}$$

The proof of (2.19) and (2.20) easily follows from the asymptotic expression (B.5) of the Bessel functions and is outlined in appendix C.

Figure 5 gives an example of the approximate equality

$$P(Q^*(t) \leqslant z) \approx F^*(z). \tag{2.21}$$

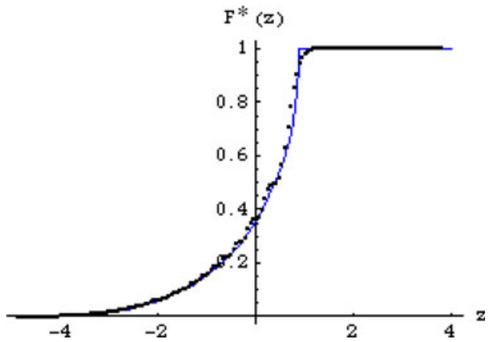


Figure 5. $F^*(z)$ (the continuous line) as an approximation, for large t, to the cumulative distribution function of $Q^*(t)$; the dots refer to the explicitly computed case $t = 100$.

We are interested in the following questions: suppose that at time $t = 0$ the cursor of a Feynman machine equipped with a semi-infinite clock is at the initial position $x = 1$; suppose the computation requires the application of T primitives and that these primitives are applied during the transitions between the sites $1, 2, \ldots, x_0 = T+1$; suppose that during the successive transitions the identity primitive is applied (the result is simply stored); suppose that, for some $x_1 > x_0$, we have a detector able to check, at any instant $t > 0$ of our choice, whether the cursor is in the assigned telomeric region $\{x_0, x_0+1, \ldots, x_1\}$; if the issue is to maximize the probability that the result of a measurement of the projector $\chi_{\{x_0,x_0+1,\ldots,x_1\}}(Q(t))$ be 1, at what time t is it most convenient to perform the measurement? How large is the maximum value attained by this probability at this optimally chosen instant?

In the case $1 \ll x_0 \ll x_1$, which requires only the consideration of large values of t, we can give an answer by an analysis of the asymptotic formula

$$P(x_0 \leqslant Q(t) \leqslant x_1) \approx F^*\left(\frac{x_1-\mu(t)}{\sigma(t)}\right) - F^*\left(\frac{x_0-\mu(t)}{\sigma(t)}\right)$$
$$= \begin{cases} \text{if} \quad x_0 < t < x_1 & 1 - \dfrac{2}{\pi}\left(\arcsin\left(\dfrac{x_0}{t}\right) - \left(\dfrac{x_0}{t}\right)\sqrt{1-\left(\dfrac{x_0}{t}\right)^2}\right) \\ \text{if} \quad t \geqslant x_1 & \dfrac{2}{\pi}\left(\arcsin\left(\dfrac{x_1}{t}\right) - \left(\dfrac{x_1}{t}\right)\sqrt{1-\left(\dfrac{x_1}{t}\right)^2}\right) \\ & -\dfrac{2}{\pi}\left(\arcsin\left(\dfrac{x_0}{t}\right) - \left(\dfrac{x_0}{t}\right)\sqrt{1-\left(\dfrac{x_0}{t}\right)^2}\right). \end{cases} \tag{2.22}$$

Figure 6 gives an example of the application of (2.22)

Equality (2.22) suggests the maximum of the right-hand side (attained, as one would have expected, at time $t = x_1$) as an upper bound for the left-hand side:

$$P(x_0 \leqslant Q(t) \leqslant x_1) \leqslant 1 - \frac{2}{\pi}\left(\arcsin\left(\frac{x_0}{x_1}\right) - \left(\frac{x_0}{x_1}\right)\sqrt{1-\left(\frac{x_0}{x_1}\right)^2}\right). \tag{2.23}$$

It is convenient to write (2.23) in terms of the length of the computation, as measured by x_0, and of the length of the telomeric chain, as measured by $\delta = x_1 - x_0$: the first term of the

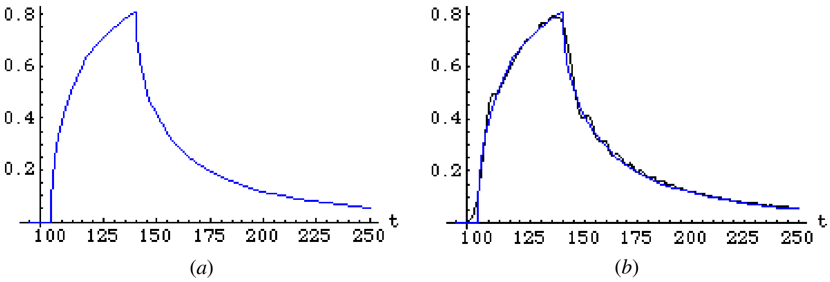


Figure 6. (*a*) The right-hand side of (2.22) computed for $x_0 = 100$ and $x_1 = 140$. (*b*) The left-hand side of (2.22) computed for the same values of x_0 and x_1: the exact result is superimposed to the approximate result.

expansion of (2.23) in the ratio $r = \frac{\delta}{x_0}$ gives the inequality (significant for $r < 0.3$)

$$P(x_0 \leqslant Q(t) \leqslant x_0 + \delta) \leqslant 1 - \frac{2}{\pi}\left(\arcsin\left(\frac{1}{1+\delta/x_0}\right) - \left(\frac{1}{1+\delta/x_0}\right)\sqrt{1-\left(\frac{1}{1+\delta/x_0}\right)^2}\right)$$

$$\leqslant \frac{4\sqrt{2}}{\pi}\sqrt{\frac{\delta}{x_0}}. \tag{2.24}$$

Note that the line of reasoning starting from (2.19) is based on the approximate treatment of $Q^*(t)$ as a continuous random variable. Therefore, our analysis does not properly cover the case, $\delta = 0$, of no telomers. However, this case can be analysed separately by considering that

$$\frac{\mathrm{d}P(Q(t) = x_0)}{\mathrm{d}t} = \frac{2x_0 J_{x_0}(t)(J_{x_0-2}(t) - J_{x_0+2}(t))}{t}. \tag{2.25}$$

Let $\tau(x_0)$ be the value of t at which $P(Q(t) = x_0)$ attains its first (and for $x_0 \gg 1$ absolute) maximum. Then $\tau(x_0)$ is the smallest positive solution of the equation

$$J_{x_0-2}(t) = J_{x_0+2}(t). \tag{2.26}$$

For large values of x_0 we have

$$\tau(x_0) \approx x_0 + c x_0^{\frac{1}{3}} \tag{2.27}$$

where ([11], p 521)

$$c \approx 0.81. \tag{2.28}$$

Therefore

$$P(Q(t) = x_0) \leqslant 4 J_{x_0}\left(x_0 + c x_0^{\frac{1}{3}}\right)^2 \leqslant \frac{4}{\pi}\sqrt{\frac{2}{c}}\frac{1}{x_0^{\frac{2}{3}}} \approx \frac{2}{x_0^{\frac{2}{3}}}. \tag{2.29}$$

In deriving (2.29), use has been made of the asymptotic expression (B.5) of $J_x(t)$ for large values of $x < t$.

Figure 7 shows that the upper bound

$$P(Q(t) = x_0) \leqslant \frac{2}{x_0^{\frac{2}{3}}} \tag{2.30}$$

becomes tight for large x_0.

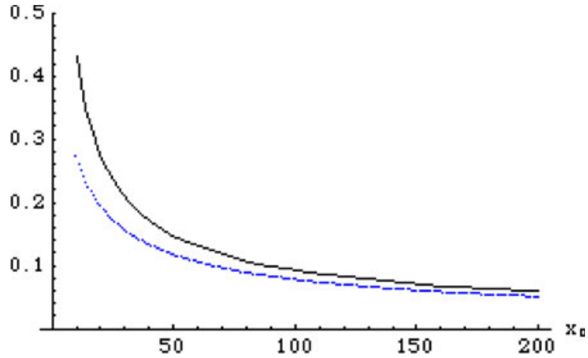


Figure 7. Comparison between $f(\tau(x_0), x_0) = P(Q(\tau(x_0)) = x_0)$ and its upper bound $2/x_0^{\frac{2}{3}}$.

3. The finite clock

The results of the previous section can be easily extended to the case of a finite clock $\Lambda_s = \{1, 2, \ldots, s\}$, by applying the Jacobi expansion [11] to the term $\exp[it\cos(\vartheta(n; s))]$ appearing in Gramß' solution:

$$\exp[it\cos\vartheta] = \sum_{k=-\infty}^{+\infty} i^k e^{ik\vartheta} J_k(t). \tag{3.1}$$

After some tedious algebra, this leads to the identity

$$\psi(t, x; s) = \varphi(t, x) + \sum_{h=1}^{\infty} (\varphi(t, 2h(s+1) + x) - \varphi(t, 2h(s+1) - x)). \tag{3.2}$$

We recall that $\psi(t, x; s)$ is, under the initial condition (2.6), the probability amplitude of finding the cursor at position x of a computational path involving a finite number s of steps (the parameter s appears in the boundary conditions (2.2) and (2.3)). The function $\varphi(t, x)$ has been defined in (2.9) as $\varphi(t, x) = \lim_{s\to+\infty} \psi(t, x; s)$ and its explicit expression in terms of Bessel functions, $\varphi(t, x) = i^{x-1}(J_{x-1}(t) + J_{x+1}(t))$, has been studied in the previous section.

Beyond the technicalities involved in (3.2), the physical idea is extremely simple: with respect to the situation considered in the previous section, in which the wave packet starting from site 1 inertially travelled towards $+\infty$, the 0 boundary condition imposed at site $s + 1$ introduces the possibility that the wave packet be reflected back to the left, with the further possibility of being scattered to the right because of the boundary condition imposed at site 0, and so on.

With the help of (3.2), we study, first of all, Feynman's original model without telomers: the number s of sites is in this case exactly the minimum number x_0 required by the computation to be performed; the computation is completed at time t, by collapse of the wavefunction, if a measurement of the projector $\chi_{\{x_0\}}(Q)$ on the state $\psi(t, x; x_0)$ gives the result 1.

This requires the study of $\psi(t, x_0; x_0)$, which, using (3.2), can be written as

$$\begin{aligned}\psi(t, x_0; x_0) &= \sum_{k=0}^{\infty} (\varphi(t, (2k+1)x_0 + 2k) - \varphi(t, (2k+1)x_0 + 2(k+1))) \\ &= (\varphi(t, x_0) - \varphi(t, x_0 + 2)) + (\varphi(t, 3x_0 + 2) - \varphi(t, 3x_0 + 4)) + \cdots. \end{aligned} \tag{3.3}$$

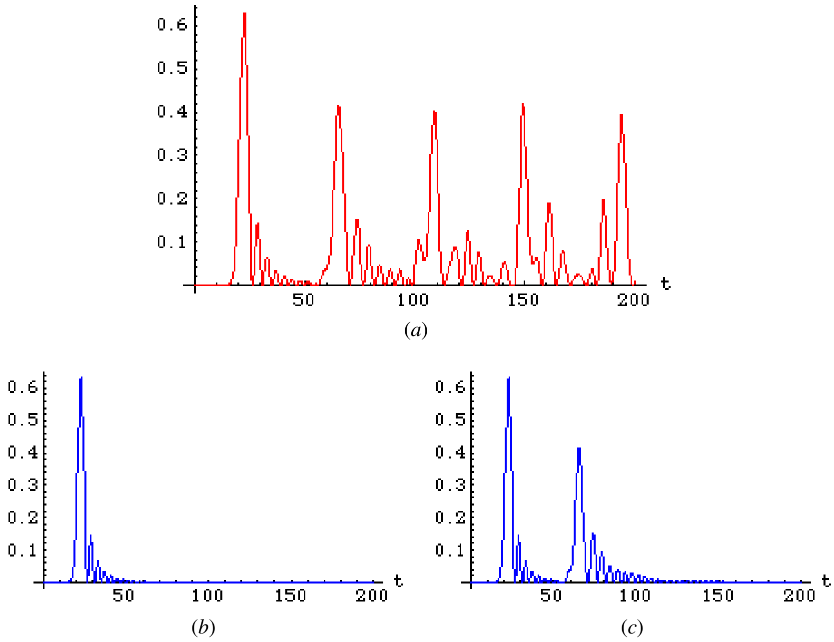


Figure 8. $x_0 = 20$: (*a*) $|\psi(t, x_0; x_0)|^2$ as a function of t; (*b*) $|\varphi(t, x_0) - \varphi(t, x_0 + 2)|^2$ as a function of t; (*c*) $|\varphi(t, x_0) - \varphi(t, x_0 + 2) + \varphi(t, 3x_0 + 2) - \varphi(t, 3x_0 + 4)|^2$.

This expansion takes into account the successive reflections of the probability amplitude due to the boundary conditions (2.2) and (2.3).

If $x_0 \gg 1$, interference effects between the direct wave $\varphi(t, x_0) - \varphi(t, x_0 + 2)$ and the successive terms, corresponding to at least one reflection at the point $x = 0$, can be neglected. In particular, the position and the height of the first maximum of $|\psi(t, x_0; x_0)|^2$ are well accounted for by the single term $(\varphi(t, x_0) - \varphi(t, x_0 + 2))$ in (3.3).

The example of figure 8 shows that additional terms are only needed to account for later and lower probability pulses. If $x_0 \gg 1$, the approximations

$$|\psi(t, x_0; x_0)|^2 \approx |\varphi(t, x_0) - \varphi(t, x_0 + 2)|^2 = |J_{x_0-1}(t) + 2J_{x_0+1}(t) + J_{x_0+3}(t)|^2 \approx 4|\varphi(t, x_0)|^2 \tag{3.4}$$

together with inequality (2.29) provide the upper bound

$$|\psi(t, x_0; x_0)|^2 \leqslant \frac{8}{x_0^{\frac{2}{3}}}. \tag{3.5}$$

By the same technique, based on (3.1), we can handle the presence of a telomeric chain extending from x_0 to $x_1 = s = x_0 + \delta$. As long as t is so small that multiple reflections can be neglected, we can write

$$\psi(t, x; x_1) \approx \varphi(t, x) - \varphi(t, 2(x_1 + 1) - x) \equiv \psi_1(t, x; x_1). \tag{3.6}$$

Figure 9 shows that the terms neglected in (3.6) account only for multiple reflections.

Setting

$$\eta(t, x) = J_{x-1}(t) + J_{x+1}(t) \tag{3.7}$$

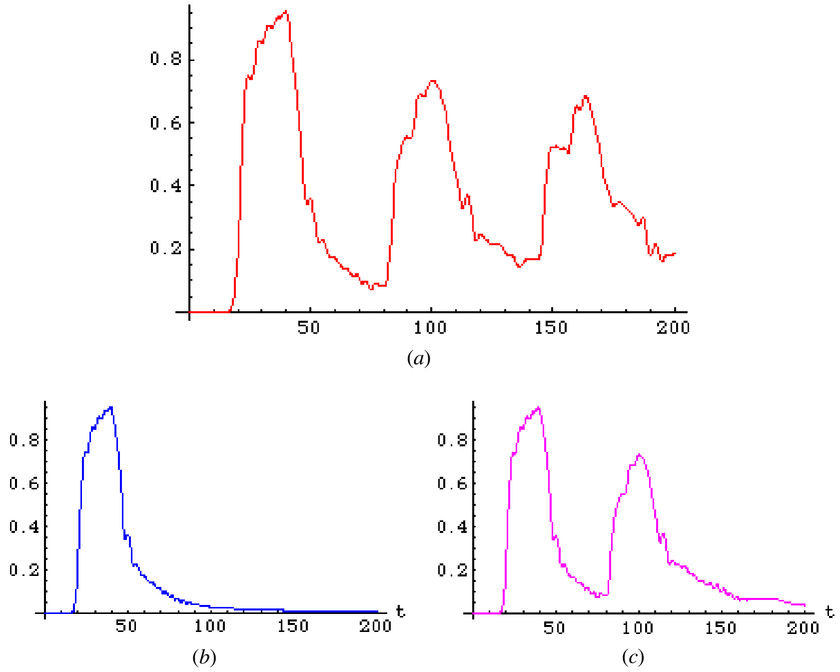


Figure 9. $x_0 = 20$, $x_1 = 30$: (*a*) $P_{x_1}(x_0 \leqslant Q(t) \leqslant x_1) \equiv \sum_{x=x_0}^{x_1} |\psi(t, x; x_1)|^2$ as a function of t; (*b*) $\sum_{x=x_0}^{x_1} |\psi_1(t, x; x_1)|^2$ as a function of t; (*c*) $\sum_{x=x_0}^{x_1} |\psi_1(t, x; x_1) + \varphi(t, 2(x_1+1)+x) - \varphi(t, 4(x_1+1)-x)|^2$ as a function of t.

for $t < 2x_1 + x_0$, we can write

$$\begin{aligned} P_{x_1}(x_0 \leqslant Q(t) \leqslant x_1) &\equiv \sum_{x=x_0}^{x_1} |\psi(t, x; x_1)|^2 \approx \sum_{x=x_0}^{x_1} |\psi_1(t, x; x_1)|^2 \\ &= \sum_{x=x_0}^{x_1} (\eta(t, x)^2 + \eta(t, 2(x_1+1) - x)^2) \\ &\quad + 2\sum_{x=x_0}^{x_1} (-1)^{x_1 - x} \eta(t, x)\eta(t, 2(x_1+1) - x) \\ &= \sum_{x=x_0}^{x_0+2(\delta+1)} \eta(t, x)^2 - \eta(t, x_1+1)^2 \\ &\quad + \sum_{\varepsilon=0}^{\delta} (-1)^{\delta-\varepsilon} \eta(t, x_0+\varepsilon)\eta(t, x_0 + 2(\delta+1) - \varepsilon). \end{aligned} \tag{3.8}$$

For $1 \ll x_0 \ll x_1$ the contribution of the last two terms in (3.8) is negligible; the first sum can, in turn, be computed by the asymptotic methods of section 2. This leads to

$$P_{x_1}(x_0 \leqslant Q(t) \leqslant x_1) \approx F^*\left(\frac{x_1 + \delta - \mu(t)}{\sigma(t)}\right) - F^*\left(\frac{x_0 - \mu(t)}{\sigma(t)}\right). \tag{3.9}$$

Figure 10 gives an idea of the approximations involved in (3.9) and of its range of validity. The maximum of the right-hand side of (3.9) is reached at $t = x_0 + 2\delta$. This leads to

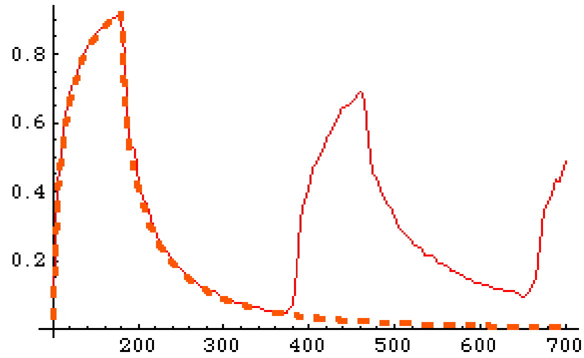


Figure 10. $x_0 = 100$, $x_1 = 140$. The thick, dashed line is the graph of the right-hand side of (3.9) as a function of t. The thin, continuous line is the graph of the left-hand side.

the upper bound

$$P_{x_1}(x_0 \leqslant Q(t) \leqslant x_0 + \delta) \leqslant 1 - \frac{2}{\pi}\left(\arcsin\left(\frac{1}{1+2\delta/x_0}\right) - \left(\frac{1}{1+2\delta/x_0}\right)\sqrt{1-\left(\frac{1}{1+2\delta/x_0}\right)^2}\right) \qquad (3.10)$$

where, we recall, $\delta = x_1 - x_0$.

The above inequality yields in turn the following:

$$P_{x_1}(x_0 \leqslant Q(t) \leqslant x_1) \leqslant \frac{8}{\pi}\sqrt{\frac{\delta}{x_0}} \qquad (3.11)$$

which is significant for $\frac{\delta}{x_0} < 0.15$.

4. Discussion

The physics behind the formalism of the two previous sections is elementary:

- The average direction of the arrow of time (as operationally defined by the mean velocity of the cursor) is determined by the spatial non-uniformity of the probability distribution of $Q(t)$. When this distribution is highly concentrated near one endpoint, it is very likely that the cursor tends to move towards the other endpoint of the available space.
- For most of the time used to complete a long computation, the cursor moves as a free Schrödinger particle, and hence is subject to spreading of the wave packet: even when the probability of finding it in the telomeric region $\{x_0, \ldots, x_1\}$ reaches its maximum value, there is a non-negligible left tail in its probability distribution, extending into the region $\{1, \ldots, x_0 - 1\}$ that corresponds to an unfinished computation.

From this point of view the standard ('static') paradigm of quantum computing appears as an idealization. According to this paradigm, the successive unitary transformations on the state of the input/output register are applied one by one by an outside agent; moreover, it is taken for granted that there is an instant at which the computation is, with certainty, completed and after which it will never be undone. With respect to Feynman's 'dynamical' paradigm, the idealization involved in the 'static' paradigm corresponds to considering the limit case $\delta \to \infty$ of an infinite amount of space available to store the result, and of an infinite amount of time available for the result to be stored with probability 1.

Whether this idealization is acceptable depends on the physical situation. Consider, for instance, the Deutsch–Joza problem [14]. As the reader will recall, the problem is to recognize whether a given Boolean function f is constant or balanced. In the static paradigm this can be done *with certainty* in one computational step by applying a unitary transformation involving, among other things, a computation of f itself. Doing the same on a Feynman machine is not an absolute impossibility because there *are* cases in which $|\psi(t,s;s)|^2$ can take the value 1, namely, the cases $s = 2$ and $s = 3$. Namely, on a Feynman machine, one can solve the Deutsch–Joza problem *with certainty* in the single step allowed if f is an 'elementary' primitive that the cursor can apply in a single transition, or in very few transitions. If many bits are involved, this amounts to assuming that one has solved the unrealistically hard problem of implementing the correct many-body interaction among the many q-bits. If, vice versa, one is forced to decompose this interaction into $x_0 \gg 1$ few-body interaction terms, one has only a probability $O\big(x_0^{-\frac{1}{2}}\big)$ of solving the problem in one step. This probability *can* be made by the addition of a suitable telomeric chain, as close to 1 as we wish; but no *finite* telomeric chain will bring it to the value 1.

Consider, from the same point of view, Grover's database search algorithm [15]: finding the single point in a Boolean function f of ν variables granted to take the value 1 requires, among other things, $2^{\frac{\nu}{2}}$ computations of f. Taking for granted that a single computation of f can be implemented as a one-step primitive on a Feynman machine: the problem remains (unless one allows for a telomeric chain of length proportional to $2^{\frac{\nu}{2}}$) that the probability of finding, upon the first measurement of the position of the cursor, that all the $x_0 = 2^{\frac{\nu}{2}}$ steps have been completed is $O\big(2^{-\frac{\nu}{4}}\big)$. Suppose one adopts the strategy of resetting the machine after each failed attempt to find the cursor in the telomeric region; according to this strategy one has to perform, before the arrival of the first success, a geometrically distributed random number of attempts, with mean value $\Omega\big(2^{\frac{\nu}{4}}\big)$, each attempt lasting a time $2^{\frac{\nu}{2}}$.

Scaling the clocking mechanism down to the quantum regime does seem, in conclusion, to be adding some additional computational costs (in terms of time, space and probability) to existing quantum algorithms.

How would these costs increase if one allowed for some random imperfections with respect to the extremely regular 'crystalline structure' of the clock considered here? Is it possible to give a quantitative assessment of Feynman's guess [2] that these imperfections would cause 'considerable havoc'?

Appendix A. Feynman's cursor model

Let N be a positive integer. Let $\mathcal{A}$ be an assigned permutation of $\{1, 2, \ldots, N\}$. Having fixed an N-dimensional Hilbert space H_N and having assigned there an orthonormal basis $|1\rangle, |2\rangle, \ldots, |N\rangle$, define a linear unitary operator A through the following action on the vectors of the assigned basis: for $y \in \{1, 2, \ldots, N\}$

$$A|y\rangle = |\mathcal{A}(y)\rangle. \tag{A.1}$$

Let $\mathcal{A} = \mathcal{A}_{s-1} \circ \cdots \circ \mathcal{A}_2 \circ \mathcal{A}_1$ be a decomposition of $\mathcal{A}$ into the product of a certain number $s-1$ of 'simpler' permutations (say, for instance, cycles of length 2). Set, for $i \in \{1, 2, \ldots, s-1\}$ and $y \in \{1, 2, \ldots, N\}$

$$A_i|y\rangle = |\mathcal{A}_i(y)\rangle. \tag{A.2}$$

It is then

$$A = A_{s-1} \cdots A_2 A_1. \tag{A.3}$$

Suppose the problem of physically implementing the action of each of the operators A_i has been solved. Feynman's cursor model addresses the issue of physically implementing the action of their product, in the correct order giving A.

This is done by considering, together with the original system with state space H_N (the 'register'), another quantum system (the 'cursor') having an s-dimensional state space H_s. Having chosen an orthonormal system $|1\rangle_c, |2\rangle_c, \ldots, |s\rangle_c$ in H_s, the time evolution of the overall system is supposed to be determined by a Hamiltonian operator of the form

$$H_{\mathcal{A}} = -\frac{K}{2}\sum_{j=1}^{s-1}(A_j \otimes |j+1\rangle_{c\ c}\langle j| + A_j^* \otimes |j\rangle_{c\ c}\langle j+1|) \tag{A.4}$$

where K is a coupling constant.

Incidentally, it is the form (A.4) of the Hamiltonian that gives a precise meaning to the requirement made above that $\mathcal{A} = \mathcal{A}_{s-1}\circ\cdots\circ\mathcal{A}_2\circ\mathcal{A}_1$ be a decomposition of $\mathcal{A}$ into the product of 'simpler' permutations: Feynman shows that in a physical implementation of register and cursor as a collection of spin-1/2 systems, one can choose this decomposition in such a way that in each addendum in H_A there appears the product of two components of spins belonging to the cursor and at most one component of a spin belonging to the register.

Consider the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle = H_{\mathcal{A}}|\phi(t)\rangle \tag{A.5}$$

under an initial condition of the particular form

$$|\phi(0)\rangle = |\phi_0\rangle \otimes |1\rangle_c \tag{A.6}$$

where $|\phi_0\rangle \in H_N$ and $|1\rangle_c \in H_s$.

It is easy to check that the form of the solution of (A.5) and (A.6) is provided by the following ansatz:

$$|\phi(t)\rangle = \sum_{k=1}^{s}\gamma(t,k;s)A_{k-1}\cdots A_1|\phi_0\rangle \otimes |k\rangle_c \tag{A.7}$$

(where, of course, the empty product of operators corresponding to $k = 1$ is to be read as the identity operator).

It is, indeed, sufficient to impose the initial condition

$$\gamma(0,k;s) = \delta_{1,k} \tag{A.8}$$

in order to satisfy (A.6), and the differential equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t,1;s) = -\frac{K}{2}\gamma(t,2;s) \tag{A.9}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t,x;s) = -\frac{K}{2}(\gamma(t,x-1;s)+\gamma(t,x+1;s)) \qquad \text{for} \quad 1 < x < s \tag{A.10}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t,s;s) = -\frac{K}{2}\gamma(t,s-1;s) \tag{A.11}$$

in order to satisfy (A.5). This fact can be easily proved using the unitarity of the operators A_i.

The main point, in the above equations, is that the evolution of the probability amplitude $\gamma(t, x; s)$ of finding the cursor at position x does not depend on the primitives A_i that it applies to the register.

Appendix B. Bessel functions of the first kind

For integer values of k, the Bessel functions of the first kind and order k, or Bessel coefficients, $J_k(t)$ are defined as the coefficients of the Laurent expansion

$$\exp\left(\frac{1}{2}t\left(w - \frac{1}{w}\right)\right) = \sum_{k=-\infty}^{+\infty} w^k J_k(t). \tag{B.1}$$

Identity (3.1) is just the above expression in the case $w = \mathrm{i}\exp(\mathrm{i}\vartheta)$.

If one takes, instead, $w = \exp(\mathrm{i}\vartheta)$ and, for a fixed integer value of x, multiplies both sides of (B.1) by $\exp(-\mathrm{i}x\vartheta)$, one obtains, by integration over $(-\pi, \pi)$, a quick (heuristic) derivation of the integral representation used in deriving (2.9):

$$J_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\mathrm{i}t\sin(\vartheta) - x\vartheta)\,\mathrm{d}\vartheta. \tag{B.2}$$

The limit relation in (2.9) is then simply, as $s \to \infty$, the convergence of the right-hand side of (2.7), seen as a Riemann sum, to an integral.

The recurrence relation used in the last step (2.9)

$$J_{x-1}(t) + J_{x+1}(t) = \frac{2x}{t}J_x(t) \tag{B.3}$$

is obtained by derivation of (B.1) with respect to w.

Derivation with respect to t gives, similarly, the recurrence relation

$$J_{x-1}(t) - J_{x+1}(t) = 2\frac{\mathrm{d}J_x(t)}{\mathrm{d}t}. \tag{B.4}$$

Figure 11 gives an idea of the behaviour of $J_x(t)$ as a function of t. Figure 12, in turn, gives an idea of the accuracy of the following asymptotic expression of the Bessel functions of large order and large argument [11, 12] used in section 2:

$$J_x(t) \approx \begin{cases} \dfrac{\exp\left(\sqrt{x^2 - t^2} - x\,\mathrm{arctanh}\left(\sqrt{1 - \frac{t^2}{x^2}}\right)\right)}{\sqrt{2\pi}(x^2 - t^2)^{1/4}} & \text{if} \quad t < x \\[2ex] \dfrac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi t^{1/3}} + \dfrac{3^{1/6}\Gamma(2/3)(t - x)}{2^{1/3}\pi t^{2/3}} & \text{if} \quad t \approx x \\[2ex] \dfrac{\sqrt{2/\pi}\cos\left(\dfrac{\pi}{4} - \sqrt{t^2 - x^2} + x\arctan\left(\sqrt{\dfrac{t^2}{x^2} - 1}\right)\right)}{(t^2 - x^2)^{1/4}} & \text{if} \quad t > x. \end{cases} \tag{B.5}$$

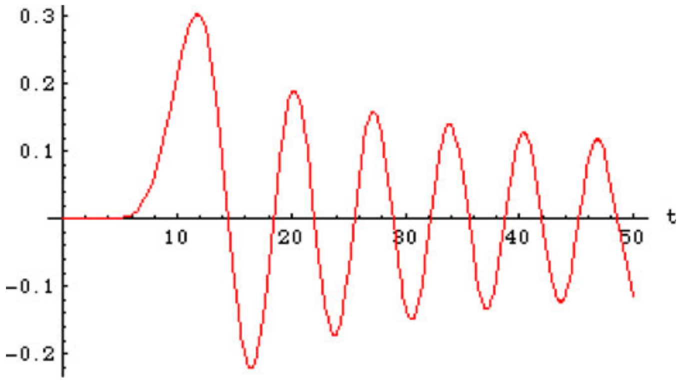


Figure 11. The graph of $J_{10}(t)$ as a function of t.

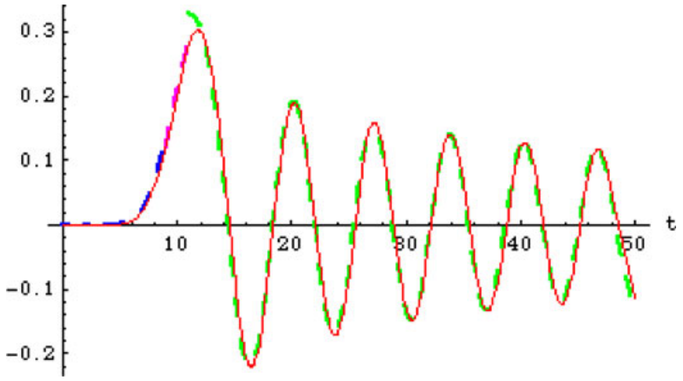


Figure 12. The right-hand side of (B.5) for $x = 10$, represented by the dashed curve, is superimposed to the graph of $J_{10}(t)$.

Appendix C. Derivation of (2.19) and (2.20)

The values that $Q^*(t)$ can take are separated by a spacing $1/\sigma(t)$ going to 0 as $t \to \infty$; it is, moreover, $Q^*(t) \geqslant (1 - \mu(t))/\sigma(t)$. We treat, then, $Q^*(t)$ as a continuous random variable having a probability density of the form

$$\rho_{Q^*}(t, z) = \begin{cases} \text{if} \quad z > (1 - \mu(t))/\sigma(t) \quad \text{then} \quad \sigma(t)4\left(\dfrac{\mu(t) + z\sigma(t)}{t}\right)^2 J_{\mu(t)+z\sigma(t)}(t)^2 \\ \text{else} \quad 0. \end{cases} \tag{C.1}$$

For $z < (t - \mu(t))/\sigma(t)$, we can use the asymptotic expression of $J_x(t)$ given by (B.5) and holding for $t > x$.

Setting

$$y(t, z) = \mu(t) + z\sigma(t) \tag{C.2}$$

we obtain, after some algebra, and neglecting a term which, for large t, is a rapidly oscillating function of z, the following approximate equality,

$$\rho_{Q^*}(t,z) \approx \frac{4\sigma(t)}{\pi t}\left(\frac{y(t,z)}{t}\right)^2 \frac{1}{\sqrt{1-\left(\frac{y(t,z)}{t}\right)^2}} \tag{C.3}$$

holding for $(1-\mu(t))/\sigma(t) < z < (t-\mu(t))/\sigma(t)$.

Setting

$$\mu_1 = \lim_{t\to\infty} \frac{\mu(t)}{t} \tag{C.4}$$

$$\sigma_1 = \lim_{t\to\infty} \frac{\sigma(t)}{t} \tag{C.5}$$

one gets, for large values of t,

$$\rho_{Q^*}(t,z) \approx \begin{cases} \text{if} \quad -\mu_1/\sigma_1 < z < (1-\mu_1/\sigma_1) \quad \text{then} \quad \dfrac{4\sigma_1}{\pi}\dfrac{(\mu_1+z\sigma_1)^2}{\sqrt{1-(\mu_1+z\sigma_1)^2}} \\ \text{else} \quad 0. \end{cases} \tag{C.6}$$

(2.19) follows from (C.6) by integration with respect to z.

(2.20) follows from the requirements that the mean and the variance of the probability density appearing on right-hand side of (C.6) be, respectively, 0 and 1.

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